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A chain of random resistors: the resistance distribution

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Abstract. The distribution of the resistance of a chain which is made of resistors with random activation energies is studied. The main purpose is to study the case of low temperature when the distribution of the resistances comprising the chain is exponentially wide and the chain resistance is typically characterized by the resistor with maximal activation energy. It appears that in this case the chain resistance distribution is so wide that the standard deviation of the resistance is much larger than the average value for the resistor with maximal activation energy. This result makes the characterization of the chain by any average resistance meaningless. The chain resistance distribution in this case appears to be not universal and strongly depends on the activation energy distribution.

1. Introduction

The main conductivity mechanisms in crystalline solids are well known and the current research in this field is concentrated on subtle details or on extreme situations (e.g., weak localization, the quantum Hall effect). In contrast, in spite of many years of investigation and a number of insights, such a clear picture does not exist for amorphous solids where carriers are strongly localized. This situation has led to a continuously growing research activity in the field.

The most popular approaches to the calculation of the resistance of amorphous solids are based on the percolation network [1]. Among the percolation models, the most realistic probably is the model with links of random resistances between the sites of the network [2]. At the present time any such link is usually envisioned as a number of resistors connected in series. The resistances of separate resistors are activation functions of temperature with randomly and independently distributed activation energies. So, given the activation energy distribution, the distribution of the resistances at low temperature is very broad. For such a broad distribution it is generally accepted that the resistance of one link is controlled by the largest resistance in the series [3] and the resistance of the percolation cluster is controlled by a critical resistance [4–7].

This picture leaves an unanswered question: to what extent is it possible to calculate the resistance of a sample if the distribution of all constituent resistances is so broad? How strongly can the resistances of two samples with different disorder realizations differ? Shapiro studied such a problem in connection with the universal scaling near the mobility edge [8]. It appeared that it was not the conductance of each sample that satisfied the universal scaling law but the conductance distribution. Raikh and Ruzin calculated the distribution function of amorphous film resistance and showed that it can be quite wide [9].

In the network of activation resistors, the situation can indeed be very dramatic, because when temperature goes to zero the width of the distribution of the constituent resistances goes

to infinity. And the purpose of this paper is to answer the question raised above in the simplest, nearly trivial case, of a one-dimensional chain of N random resistors.

This answer is important for all systems where the transport can be considered as one-dimensional or quasi-one-dimensional. These can be narrow-channel metal–oxide–semiconductor field-effect transistors which have been shown to be effectively one dimensional [10] or thin insulating films where the transport is dominated by rare one-dimensional chains with relatively large conductance [11–13]. Chains of resistors can also be good models for polymers. The importance of the problem goes beyond one-dimensional systems, because a one-dimensional chain can be considered as the simplest model of a link between sites of a multi-dimensional network.

The study of the one-dimensional problem seems to be trivial. Indeed, the resistance of the chain equals the sum of all constituent resistances and an analytic expression for the probability distribution of the resistance can be immediately written down. However, the reduction of the probability distribution of the chain resistance to a tractable form is not simple. The difficulty appears when the number of resistors in the chain (N) is large, which is the most interesting case. For one specific case the distribution was studied by Raikh and Ruzin [9]. The purpose of the present paper is to develop a general approach and to study a few of the most interesting examples of the activation energy distribution. The result shows that when the temperature is so small that the resistance of the chain is controlled by just one resistor, the resistance distribution is not universal and its shape strongly depends on the distribution of the activation energies of the separate constituent resistors (hereafter they will be called partial resistors). Typically, the resistance distribution becomes so broad that the average resistance does not make any sense; it is much smaller than the mean square deviation from it.

The paper is organized in the following way. After all necessary definitions are introduced in section 2, possible asymptotic cases and related simplifications are considered in section 3. The main problem in this consideration is the interplay between two dimensionless parameters which strongly affects the chain resistance distribution. One parameter is the ratio of the width of the activation energy distribution and temperature while the other is the number of partial resistors. In section 4 the results obtained in the previous section are applied to some of the most often encountered activation energy distributions. The last section contains a discussion of the results.

2. Probability distribution of the chain resistance

The resistance R of the chain that consists of N resistors with the resistances $r_0 e^{\epsilon_k/T}$ is just the sum of those resistances:

$$R = r_0 \sum_{k=1}^N e^{\epsilon_k/T}. \quad (2.1)$$

If the probability distributions $p(\epsilon)$ of all activation energies ϵ_k are the same, then the probability distribution of the total resistance $P(R)$ can be calculated with the help of the characteristic function:

$$P(R) = \frac{1}{2\pi} \int e^{-iRt} F(t) dt \quad (2.2)$$

where the global characteristic function $F(t)$ is connected in a simple way to the partial characteristic function $f(t)$:

$$F(t) = f^N(t) \quad (2.3a)$$

$$f(t) = \int \exp(itr_0 e^{\epsilon/T}) p(\epsilon) d\epsilon. \quad (2.3b)$$

In the calculation of $F(t)$, sometimes it is important to remember that because $r_0 > 0$ the integral (2.3b) is convergent for $\text{Im } t > 0$. As a result $f(t)$ is regular in the upper half of the complex plane t . This property makes the integral in equation (2.2) equal to zero for $R < 0$, i.e., the probability of finding a negative resistance equals zero.

The characteristic function can be expanded in the momenta of $P(R)$:

$$F(t) = \sum_{k=0}^{\infty} \frac{(it)^k}{k!} \overline{R^k} \tag{2.4}$$

where

$$\overline{R^k} = \int P(R) R^k dR. \tag{2.5}$$

With the help of equation (2.3), these momenta can be expressed in terms of the averaged powers of partial resistances:

$$\overline{R^k} = \sum_{\substack{l_1+2l_2+3l_3+\dots=k \\ l_0+l_1+l_2+\dots=N}} \frac{N!k!}{l_0!l_1!l_2!l_3!\dots} \left(\frac{\bar{r}}{1!}\right)^{l_1} \left(\frac{\bar{r}^2}{2!}\right)^{l_2} \left(\frac{\bar{r}^3}{3!}\right)^{l_3} \dots \tag{2.6}$$

where

$$\overline{r^k} = r_0^k \int e^{k\epsilon/T} p(\epsilon) d\epsilon. \tag{2.7}$$

3. Main extreme cases

3.1. Universal Gaussian distribution

The simplest and most well known distribution of the resistance is obtained if the temperature is not very small compared to the width of $p(\epsilon)$, E . In this case many resistors contribute to the resistance of the chain and the resistance distribution $P(R)$ is Gaussian. Hereafter it will be called the universal Gaussian distribution to distinguish it from the Gaussian distribution of the activation energies $p(\epsilon)$. Formally, it is obtained from the exponential expansion

$$F(t) = e^{N \ln f(t)} = \exp\left(N \sum_{k=1}^{\infty} c_k (it)^k\right) \tag{3.1}$$

where

$$c_k = \sum_{l_1+2l_2+3l_3+\dots} (-1)^{l_1+l_2+\dots-1} \frac{(l_1+l_2+\dots-1)!}{l_1!l_2!l_3!\dots} \left(\frac{\bar{r}}{1!}\right)^{l_1} \left(\frac{\bar{r}^2}{2!}\right)^{l_2} \left(\frac{\bar{r}^3}{3!}\right)^{l_3} \dots \tag{3.2}$$

If in the exponent in equation (3.1) all terms containing t^k with $k > 2$ are neglected, i.e.,

$$F(t) = \exp\left[iNt\bar{r} - \frac{Nt^2}{2}(\bar{r}^2 - \bar{r}^2)\right] \tag{3.3}$$

then

$$P(R) = \frac{1}{\sqrt{2\pi(\Delta R)^2}} e^{-(R-\bar{R})^2/2(\Delta R)^2} \tag{3.4}$$

where $\bar{R} = N\bar{r}$ and $(\Delta R)^2 = N(\bar{r}^2 - \bar{r}^2)$.

The criterion for the validity of equation (3.3) can be obtained in the following way. When equation (3.3) is substituted in equation (2.2), the main contribution to the integral comes from

$t \sim 1/\sqrt{N(\bar{r}^2 - \bar{r}^2)}$. The result is valid if in this region of the integration all neglected terms are small, i.e., $c_k/[N(\bar{r}^2 - \bar{r}^2)]^{k/2} \ll 1$. It is enough to satisfy this criterion for $k = 3$, which gives

$$\frac{\bar{r}^3 - 3\bar{r}^2\bar{r} + 2\bar{r}^3}{6(\bar{r}^2 - \bar{r}^2)^{3/2}} \ll N^{1/2}. \tag{3.5}$$

3.2. *Low temperature*

When the temperature is small compared to the width of the energy distribution:

$$T \ll E \tag{3.6}$$

the contribution to R of the resistors with small activation energy can be neglected. In other words, one can expect only a high-energy tail of $p(\epsilon)$ to be important for the calculation of $P(R)$. The situation is, however, complicated by the fact that the number of resistors with high activation energies is relatively small and when the activation energy goes down the growth of the number of resistors can compete with the reduction of their resistance. The only simplification that can be made under the condition (3.6) is that the characteristic scale of R (which can be the width of $P(R)$) is much larger than the average partial resistance. This means that partial characteristic function (2.3b)

$$f(t) = 1 + \frac{itr_0}{1!} \int e^{\epsilon/T} p(\epsilon) d\epsilon + \frac{(itr_0)^2}{2!} \int e^{2\epsilon/T} p(\epsilon) d\epsilon + \dots \tag{3.7}$$

is very close to unity because the characteristic values of t are of the order of the characteristic $1/R$. As a result, the global characteristic function (2.3a)

$$F(t) = \exp \left\{ N \ln \left[1 - \int (1 - e^{itr_0 \exp(\epsilon/T)}) p(\epsilon) d\epsilon \right] \right\} \\ \approx \exp \left\{ -N \int [1 - e^{itr_0 \exp(\epsilon/T)}] p(\epsilon) d\epsilon \right\}. \tag{3.8}$$

The same expression can be obtained in a more intuitive way. Under the condition (3.6), the chain resistance R can be approximated by the sum of the resistors with activation energies larger than some value ϵ . The probability distribution of n activation energies $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ such that $\epsilon_1 > \epsilon_2 > \dots > \epsilon_n > \epsilon$ is

$$p_{Mn}(\epsilon_1, \epsilon_2, \dots, \epsilon_n; \epsilon) = \frac{N!}{(N-n)!} w^{N-n}(\epsilon) \theta(\epsilon_n - \epsilon) \left[\prod_{j=1}^{n-1} \theta(\epsilon_j - \epsilon_{j+1}) \right] \left[\prod_{j=1}^n p(\epsilon_j) \right] \tag{3.9}$$

where

$$w(\epsilon) = \int_{-\infty}^{\epsilon} p(\epsilon') d\epsilon' \tag{3.10}$$

is the probability of the activation energy being smaller than ϵ . The integration of p_{Mn} with respect to $\epsilon_1, \epsilon_2, \dots, \epsilon_n$ results in the binomial distribution of $N - n$ energies in the interval $(-\infty, \epsilon)$ and n energies in the interval (ϵ, ∞) :

$$\int p_{Mn}(\epsilon_1, \epsilon_2, \dots, \epsilon_n; \epsilon) d\epsilon_1 d\epsilon_2 \dots d\epsilon_n = \frac{N!}{(N-n)!n!} w^{N-n}(\epsilon) [1 - w(\epsilon)]^n \\ \approx \frac{\bar{n}^n(\epsilon)}{n!} e^{-\bar{n}(\epsilon)}. \tag{3.11}$$

The number of resistors contributing to R is much smaller than N , so the binomial distribution can be approximated with the Poissonian one, where $\bar{n}(\epsilon) = N[1 - w(\epsilon)]$ is the average number of activation energies larger than ϵ .

Under the same condition, the characteristic function

$$\begin{aligned}
 F(t) &= \sum_{n=0}^N \int \exp\left(itr_0 \sum_{j=1}^n e^{i\epsilon_j/T}\right) p_{Mn}(\epsilon_1, \epsilon_2, \dots, \epsilon_n; \epsilon) d\epsilon_1 d\epsilon_2 \cdots d\epsilon_n \\
 &\approx e^{-\bar{n}(\epsilon)} \sum_{n=0}^{\infty} \frac{N^n}{n!} \int_{\epsilon}^{\infty} \exp\left(itr_0 \sum_{j=1}^n e^{i\epsilon_j/T}\right) \prod_{l=1}^n p(\epsilon_l) d\epsilon_l \\
 &= \exp\left\{-N \int_{\epsilon}^{\infty} [1 - e^{itr_0 \exp(\epsilon'/T)}] p(\epsilon') d\epsilon'\right\}
 \end{aligned} \tag{3.12}$$

which is reduced to equation (3.8) when ϵ goes to $-\infty$.

For a large enough N and low temperature, both conditions (3.6) and (3.5) are satisfied. In this region the integral in the exponent in equation (3.8) can be expanded in t , which leads eventually to the universal Gaussian distribution. The only difference between this distribution and equation (3.3) is that in the low-temperature case \bar{r}^2 is neglected compared to r^2 . Under the condition (3.6), the relation $\bar{r}^2 \ll r^2$ is true for any continuous and smooth enough energy distribution.

3.3. Extremely low temperature

For an extremely low temperature the difference between the largest activation energy and the next one to it, Δ , is larger than the temperature. Under this condition, the value of the chain resistance is exponentially close to the value of the largest partial resistance. The probability distribution of the largest activation energy is

$$p_{M1}(\epsilon) = Ne^{-\bar{n}(\epsilon)} p(\epsilon). \tag{3.13}$$

This immediately gives the characteristic function

$$F(t) = N \int e^{-\bar{n}(\epsilon) + itr_0 \exp(\epsilon/T)} p(\epsilon) d\epsilon \tag{3.14}$$

and the distribution function

$$P(R) = \frac{T}{R} p_{M1}\left(T \ln \frac{R}{r_0}\right). \tag{3.15}$$

To find the region of parameters where equation (3.15) is justified, it is convenient to start with the probability distributions of the two largest activation energies ϵ and $\epsilon - \Delta$ ($\Delta > 0$):

$$p_{M2}(\epsilon, \epsilon - \Delta) = N(N - 1)w^{N-n}(\epsilon - \Delta)p(\epsilon)p(\epsilon - \Delta) \approx e^{-\bar{n}(\epsilon - \Delta)} N^2 p(\epsilon)p(\epsilon - \Delta). \tag{3.16}$$

With the help of this expression, the probability distribution of the difference Δ assuming $\Delta \ll \epsilon$ is

$$p_D(\Delta) = \int p_{M2}(\epsilon, \epsilon - \Delta) d\epsilon = N^2 \int e^{-\bar{n}(\epsilon) - N\Delta p(\epsilon)} p^2(\epsilon) d\epsilon. \tag{3.17}$$

This means that $\Delta \sim 1/Np(\epsilon_m)$ where ϵ_m is the most probable value of the maximal energy (this is the point where the most important factor in the integrand in equation (3.17), namely $p(\epsilon)e^{-\bar{n}(\epsilon)}$, has a maximum). The latter can be found from the equation

$$Np^2(\epsilon_m) + p'(\epsilon_m) = 0. \tag{3.18}$$

So equation (3.15) is justified if

$$T \ll 1/Np(\epsilon_m). \quad (3.19)$$

Practically, this criterion is the opposite to equation (3.5). This can be easily seen for all $p(\epsilon)$ considered in section 4.

Generally speaking, the temperature region where inequality (3.19) is satisfied is a part of the region limited by the condition (3.6). This means that the distribution (3.15) is a particular case of a more general distribution obtained from the characteristic function (3.8). The corresponding derivation is given in appendix A.

4. Some specific activation energy distributions

In this section the chain resistance distribution function is calculated for some simple and often used activation energy distributions. The simplest of them is the discrete distribution where the activation energy can take values $\pm E$. A similar distribution is sometimes used in percolation problems [14]. Then the uniform distribution where the energy can take any value in some interval with equal probability is considered. The uniform distribution is often used in simulations related to the Coulomb gap problem [1]. Then also the Gaussian distribution and exponential distribution are studied. The Gaussian distribution is encountered so often that it does not need any comment. The exponential distribution also appears in practice and it is particularly interesting because the existence of r^k in this case depends on temperature. Finally the Lorentzian distribution is considered. This distribution appears in some glass problems [15, 16] and is an example of a distribution with long tails for which no momentum exists.

4.1. Discrete distribution

Probably the simplest example of a random distribution is

$$p(\epsilon) = w\delta(\epsilon + E) + (1 - w)\delta(\epsilon - E) \quad (4.1)$$

where $w < 1$. This immediately gives

$$P(R) = \sum_{k=0}^N \frac{N!}{k!(N-k)!} w^k (1-w)^{N-k} \delta(R - kr_0 e^{-E/T} - (N-k)r_0 e^{E/T}). \quad (4.2)$$

When $N \gg 1$ the maximum contribution to the sum comes from k close to that which gives the maximum to the coefficient for the δ -function, and equation (4.2) can be reduced to

$$P(R) = \sum_j \frac{1}{\sqrt{2\pi N w(1-w)}} e^{-j^2/2Nw(1-w)} \times \delta(R - (wN + j)r_0 e^{-E/T} - (N - wN - j)r_0 e^{E/T}). \quad (4.3)$$

This distribution is discrete. At high temperature, $T \gg E$, the discreteness is not significant. If it is neglected, the sum in equation (4.3) can be replaced with an integral which results in the universal Gaussian distribution (3.3) with parameters $\bar{R} = Nr_0[1 + (1 - 2w)E/T]$ and $(\Delta R)^2 = 4Nr_0^2 w(1-w)(E/T)^2$.

At low temperature, $T \ll E$, the contribution of resistances with the activation energy $-E$ can be neglected. The distribution (4.2) in this case is strongly discrete and equations (3.12) and (3.15) give quite a rough approximation.

4.2. Uniform distribution

The uniform distribution is

$$p(\epsilon) = \begin{cases} \frac{1}{2E} & |\epsilon| < E \\ 0 & |\epsilon| > E. \end{cases} \quad (4.4)$$

The average powers of the partial resistances in this case are

$$\overline{r^k} = r_0^k \frac{T}{kE} \sinh \frac{kE}{T}. \quad (4.5)$$

The universal Gaussian distribution has parameters

$$\overline{R} = r_0 \frac{NT}{E} \sinh \frac{E}{T} \quad (4.6a)$$

$$\overline{(\Delta R)^2} = r_0^2 \frac{NT}{2E} \left(\sinh \frac{2E}{T} - \frac{2T}{E} \sinh^2 \frac{E}{T} \right). \quad (4.6b)$$

According to equation (3.5), this distribution is valid when $N \gg 1 + (E/T)$.

At low temperature, when $E \gg T$ but the condition $N \gg E/T$ is still satisfied, equation (4.6) is simplified and the universal Gaussian distribution is equivalent to the distribution of the chain of equal resistances, $(r_0/2)e^{E/T}$, whose number has a Poissonian distribution with the average value NT/E .

In general, at low temperature, $E \gg T$, the distribution function can be reduced to the form

$$P(R) = \frac{1}{2\pi} \int \exp \left[-itR - \frac{NT}{2E} \int_0^{ir_M} (1 - e^{iu}) \frac{du}{u} \right] dt \quad (4.7)$$

where $r_M = r_0 e^{E/T}$ is the maximal partial resistance.

The average separation between activation energies is E/N and further simplification of the distribution function can be achieved at temperatures smaller than this value:

$$E \gg NT. \quad (4.8)$$

This condition is opposite to the criterion for the validity of the universal Gaussian distribution at low temperature. Then equation (3.15) gives

$$P(R) = \frac{NT}{2ER} \left(\frac{r_M}{R} \right)^{NT/2E} \quad (4.9)$$

where $R < r_M$. This distribution is characterized by the average value $\overline{R} = Nr_M(T/2E)$ and the standard deviation

$$\sqrt{\overline{(\Delta R)^2}} = r_M \sqrt{NT/4E}.$$

That is, the width of the distribution is larger than the average resistance by the factor $(E/NT)^{1/2} \gg 1$.

4.3. Gaussian distribution

The Gaussian distribution is

$$p(\epsilon) = \frac{1}{\sqrt{2\pi}E} e^{-\epsilon^2/2E^2} \quad (4.10)$$

with $\overline{\epsilon^2} = E^2$. The average powers of the partial resistances

$$\overline{r^k} = r_0^k e^{k^2 E^2 / 2T^2}. \quad (4.11)$$

This expression shows that the momenta of the chain resistance distribution, $\overline{R^k}$, all exist but grow with k so fast that the series (2.4) for the characteristic function diverges.

Under the condition (3.5) which now takes the form $N \gg 1 + e^{3E^2/2T^2}$, $P(R)$ becomes the universal Gaussian distribution with the parameters

$$\overline{R} = Nr_0 e^{E^2/2T^2} \quad (4.12a)$$

$$(\Delta R)^2 = Nr_0^2 (e^{2E^2/T^2} - e^{E^2/T^2}). \quad (4.12b)$$

At low temperature, $T \ll E$, the characteristic function (3.8) contains complicated integrals which, in general, cannot be simplified. For this reason it makes sense to consider only the case of extremely low temperature.

According to equation (3.18), the most probable value of the maximal activation energy $\epsilon_m = x_m E$ where x_m has to be found from the equation

$$x_m = \frac{N}{\sqrt{2\pi}} e^{-x_m^2/2} \quad (4.13)$$

which gives $x_m \approx \sqrt{2 \ln N}$. So the condition for extremely low temperature (3.19) is

$$T \ll E/x_m \approx E/\sqrt{\ln N}. \quad (4.14)$$

The average number of activation energies above the value of $\epsilon \gg E$ is

$$\bar{n}(\epsilon) = \frac{N}{\sqrt{2\pi} E} \int_{\epsilon}^{\infty} e^{-x^2/2E^2} dx \approx \frac{NE}{\sqrt{2\pi} \epsilon} e^{-\epsilon^2/2E^2} \quad (4.15)$$

and the distribution of the maximal activation energy (3.13) becomes

$$p_{M1}(\epsilon) = \frac{N}{\sqrt{2\pi} E} \exp\left(-\frac{NE}{\sqrt{2\pi} \epsilon} e^{-\epsilon^2/2E^2} - \frac{\epsilon^2}{2E^2}\right). \quad (4.16)$$

This function has a sharp maximum at $\epsilon = \epsilon_m$. Let $\delta\epsilon = \epsilon - \epsilon_m$; then in the region $|\delta\epsilon| \ll E$,

$$p_{M1}(\epsilon) = \frac{x_m}{E} \exp\left(-e^{-x_m \delta\epsilon/E} - \frac{x_m \delta\epsilon}{E}\right). \quad (4.17)$$

It is easy to see that the normalization integral of this distribution converges in the region $|\delta\epsilon| \sim E/x_m \ll E$, which means that the expansion in the region $|\delta\epsilon| \ll E$ is justified.

Thus, under the condition (4.14), equation (3.15) results in

$$P(R) = \frac{1}{R} \frac{x_m T}{E} \left(\frac{r_M}{R}\right)^{x_m T/E} e^{-(r_M/R)^{x_m T/E}} \quad (4.18)$$

where $r_M = r_0 e^{\epsilon_m/T}$.

This distribution gives

$$\overline{R} = r_M \Gamma\left(\frac{E}{Tx_m} + 1\right) \approx r_M \sqrt{2\pi} \left(\frac{E}{Tx_m}\right)^{E/Tx_m + 1/2} e^{-E/Tx_m} \quad (4.19)$$

and the most probable value

$$R_{P=max} = r_M \left[\frac{x_m T/E}{1 + (x_m T/E)} \right]^{E/x_m T} \approx \frac{r_M}{e} \left(\frac{x_m T}{E}\right)^{E/x_m T}. \quad (4.20)$$

First of all it is interesting to note that the average resistance and the most probable resistance are of different order, $\overline{R} \gg R_{P=max}$. This results from the large width of the distribution (4.18). This distribution is so broad that its second moment, $\overline{R^2}$, does not exist.

These results can be compared with the exact expression for \overline{R} and $(\Delta R)^2$ for $T \ll E$, equation (4.12). Under the condition (4.14), the value of \overline{R} obtained from equation (4.12a) is much larger than that obtained from equation (4.19). This also is an indication that the width of the exact distribution is much larger than the width of its maximum. A very large width is manifested also in the inequality $(\Delta R)^2 \gg (\overline{R})^2$ that follows from equation (4.12) for $T \ll E$.

4.4. Exponential distribution

The exponential distribution is defined as

$$p(\epsilon) = \frac{1}{2E} e^{-|\epsilon|/E}. \tag{4.21}$$

Moments of partial resistances exist only for high enough temperature:

$$\overline{r^k} = r_0^k \frac{T^2}{T^2 - k^2 E^2} \tag{4.22}$$

for $T > kE$.

The universal Gaussian distribution (3.4) exists only for $T > 2E$ and has parameters

$$\overline{R} = Nr_0 \frac{T^2}{T^2 - E^2} \tag{4.23a}$$

$$\overline{(\Delta R)^2} = Nr_0^2 \frac{T^2 E^2 (2T^2 + E^2)}{(T^2 - E^2)^2 (T^2 - 4E^2)}. \tag{4.23b}$$

According to equation (3.5), the distribution is definitely valid when $T > 3E$, and $\overline{r^3}$ does exist. One can hope, however, that for its validity the existence of $\overline{r^2}$, i.e., $T > 2E$, is enough. This can indeed be proved with the help of the study of

$$f(t) = \frac{T}{2E} \left[\int_0^1 e^{itr_0 x} x^{(T/E)-1} dx + \phi(tr_0) \right] \tag{4.24a}$$

$$\phi(t) = \int_1^\infty e^{itx} x^{-(T/E)-1} dx \tag{4.24b}$$

except for a very narrow region of temperature near $2E$. In the close vicinity of $2E$, the distribution deviates from the universal Gaussian one.

Even when the universal Gaussian distribution is not valid it is possible to use the same approach as led to equation (3.3) in order to find the true distribution. The point is that when the temperature is not small compared to E , then since $N \gg 1$ only small values of t contribute to the integral in equation (2.2). So it is necessary to find the behaviour of $f(t)$ at small t . The first term in equation (4.24a) is a regular function of t and only $\phi(t)$ presents a little problem. In the region $2E > T > E$,

$$\phi(t) = \frac{E}{T} + \frac{E}{T} \left(\frac{E}{T-E} - 1 \right) it + \frac{E^2}{T(T-E)} |t|^{T/E} e^{-i(\pi T/2E)\text{sgn}(t)} \Gamma(2 - T/E) \tag{4.25}$$

which leads to

$$P(R) = \frac{1}{\pi \Delta R} Q_{T/E} \left(\frac{R - R_0}{\Delta R} \right) \tag{4.26}$$

where

$$Q_\nu(\rho) = \text{Re} \int_0^\infty e^{-i\rho t + t^\nu e^{-i\pi\nu/2}} dt \tag{4.27}$$

and

$$R_0 = \frac{NE^2}{T^2 - E^2} r_0 \tag{4.28a}$$

$$\Delta R = \left| \frac{N}{2} \Gamma(1 - T/E) \right|^{E/T} r_0. \tag{4.28b}$$

If $T < E$, then in the expansion of $\phi(t)$ only terms of the lowest order in t have to be kept, i.e.,

$$\phi(t) = \frac{E}{T} - \frac{E}{T} |t|^{T/E} e^{-i\pi(T/2E)\text{sgn}(t)} \Gamma(1 - T/E) \quad (4.29)$$

which results in

$$P(R) = \frac{1}{\pi \Delta R} Q_{T/E} \left(\frac{R}{\Delta R} \right). \quad (4.30)$$

In the regions where the temperature is very close to $2E$ or E , expressions (4.26) and (4.30) have also some logarithmic corrections.

If $T \ll E$, equation (4.30) is still valid and equation (4.27) can be simplified, so

$$P(R) = \frac{T}{R} \frac{N}{2E} \exp \left[-\frac{N}{2} \left(\frac{r_0}{R} \right)^{T/E} \right] \left(\frac{r_0}{R} \right)^{T/E}. \quad (4.31)$$

The same result can be obtained with the help of equation (3.15). The most probable maximal activation energy for the exponential distribution is $\epsilon_m = E \ln(N/2)$, and equation (3.19) gives for this result the same criterion, $T \ll E$.

The most probable resistance of the distribution (4.31) is

$$R_{P=\max} = r_0 \left[\frac{NT/E}{2(1+T/E)} \right]^{E/T} \approx \frac{r_0}{e} \left(\frac{NT}{2E} \right)^{E/T}. \quad (4.32)$$

No momenta of this distribution exist, which is an indication that it is very broad and deviations from the most probable value can be very big.

4.5. Lorentzian distribution

The Lorentzian distribution:

$$p(\epsilon) = \frac{1}{\pi} \frac{E}{\epsilon^2 + E^2} \quad (4.33)$$

is so broad that all its momenta are infinite. For this reason it does not lead to the universal Gaussian distribution when $T \gtrsim E$. But at high temperature, $T \gg E$,

$$f(t) = \frac{1}{\pi} \int \exp [itr_0 e^{(E/T)x}] \frac{dx}{x^2 + 1} \approx \frac{1}{\pi} \int e^{itr_0[1+(E/T)x]} \frac{1}{x^2 + 1} dx = e^{itr_0 - |t|r_0(E/T)}. \quad (4.34)$$

As a result the chain resistance distribution:

$$P(R) = \frac{1}{\pi} \frac{Nr_0(E/T)}{(R - Nr_0)^2 + (Nr_0E/T)^2} \quad (4.35)$$

is the Lorentzian distribution.

At low temperature:

$$NE \gg T \quad (4.36)$$

it is convenient to make use of the probability distribution of the maximal energy,

$$p_m(\epsilon) = \frac{NE}{\pi\epsilon^2} e^{-NE/\pi\epsilon} \theta(\epsilon) \quad (4.37)$$

and equation (3.15) gives

$$P(R) = \frac{NE}{\pi RT \ln^2(R/r_0)} \exp \left[-\frac{NE}{\pi T \ln(R/r_0)} \right] \theta(R - r_0). \quad (4.38)$$

This distribution also has such long tails that it has no moment.

5. Conclusions

In previous sections, the distribution of the chain of resistors with random activation energies has been calculated for different temperature regions. The simplest situation appears at temperature larger than the width of the activation energy distribution. Then, many resistors contribute to the chain resistance and the distribution of the chain resistance is Gaussian.

In the most interesting case, when the temperature is so small that the total resistance is controlled by only one resistor with maximal activation energy, the situation is quite different. The resistance distribution in this case is so wide that its breadth is larger than the average chain resistance. This makes the characterization of a chain by its average resistance meaningless. According to equation (3.15), the distribution of the resistance logarithm (compare reference [9]) is not that wide, but the measured quantity is still the resistance. Another important feature is that in the low-temperature case the chain resistance distribution is not universal and strongly depends on the activation energy distribution. For example, for the Lorentzian activation energy distribution, even the average value of the chain resistance logarithm does not exist.

Appendix A. Expansion in the number of high activation energies

It is possible to obtain an expansion for the characteristic function (3.8) in the number of resistors with highest activation energies that eventually leads to equation (3.14). For this purpose it is convenient first to present the expression for $F(t)$ in a form that emphasizes the importance of high activation energies, namely,

$$F(t) = 1 + N \int_{-\infty}^{\infty} \exp \left[-\bar{n}(\epsilon) + N \int_{\epsilon}^{\infty} e^{itr_0 \exp(\epsilon'/T)} p(\epsilon') d\epsilon' \right] \left[e^{itr_0 \exp(\epsilon/T)} - 1 \right] p(\epsilon) d\epsilon. \tag{A.1}$$

The factor $e^{-\bar{n}(\epsilon)} p(\epsilon)$ concentrates the integrand in the region close to the highest activation energy $\epsilon_m \gtrsim E$. For $T \ll E$ the oscillating function in the integrand of the integral with respect to ϵ' limits the integration to a region of the order of T and a good estimate of the integral in the exponent is

$$N \int_{\epsilon}^{\infty} e^{itr_0 \exp(\epsilon'/T)} p(\epsilon') d\epsilon' = N e^{itr_0 \exp(\epsilon/T)} \int_0^{\infty} e^{itr_0 \exp(x/T)} p(\epsilon + x) dx \sim NT p(\epsilon). \tag{A.2}$$

So, under the condition (3.19), the integrand in equation (A.1) can be expanded in the integral in the exponent:

$$F(t) = 1 + \sum_{n=1}^{\infty} \frac{N^n}{(n-1)!} \int_{-\infty}^{\infty} e^{-\bar{n}(\epsilon)} p(\epsilon) d\epsilon \left[\int_{\epsilon}^{\infty} e^{itr(\epsilon')} p(\epsilon') d\epsilon' \right]^{n-1} \left[e^{itr(\epsilon)} - 1 \right]. \tag{A.3}$$

This series can be transformed with the help of the identity

$$F_n(t) = F_{n-1}(t) + \frac{N^n}{(n-1)!} \int_{-\infty}^{\infty} e^{-\bar{n}(\epsilon)} p(\epsilon) d\epsilon \left[\int_{\epsilon}^{\infty} e^{itr(\epsilon')} p(\epsilon') d\epsilon' \right]^{n-1} \left[e^{itr(\epsilon)} - 1 \right] \tag{A.4}$$

which can be obtained by integration by parts and where

$$F_n(t) = \frac{N^n}{(n-1)!} \int_{-\infty}^{\infty} e^{-\bar{n}(\epsilon)} p(\epsilon) d\epsilon \left[\int_{\epsilon}^{\infty} e^{itr(\epsilon')} p(\epsilon') d\epsilon' \right]^{n-1} \tag{A.5}$$

and the term of the order of e^{-N} is neglected.

The identity (A.4) gives

$$F(t) = \lim_{n \rightarrow \infty} F_n(t). \quad (\text{A.6})$$

On the other hand, under the condition (3.19), $F(t) \approx F_n(t)$ with a small value of n . For a finite value of n , the expression (A.5) for $F_n(t)$ can be written as the characteristic function of the distribution of the sum of n resistances with largest activation energies:

$$F_n(t) = \int e^{itr_0 \sum_{j=1}^n \exp(\epsilon_j/T)} p_{Mn}(\epsilon_1, \epsilon_2, \dots, \epsilon_n) \prod_{j=1}^n d\epsilon_j \quad (\text{A.7})$$

where

$$p_{Mn}(\epsilon_1, \epsilon_2, \dots, \epsilon_n) = e^{-\bar{n}(\epsilon_n)} N^n \left[\prod_{j=1}^{n-1} \theta(\epsilon_j - \epsilon_{j+1}) \right] \left[\prod_{j=1}^n p(\epsilon_j) \right] \quad (\text{A.8})$$

is the probability distribution of the n largest activation energies (compare equation (3.9)). For $n = 1$, equation (A.8) becomes (3.14).

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